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Generation and monitoring of discrete stable random processes using multiple immigration population models

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Abstract

Some properties of classical population processes that comprise births, deaths and multiple immigrations are investigated. The rates at which the immigrants arrive can be tailored to produce a population whose steady state fluctuations are described by a pre-selected distribution. Attention is focused on the class of distributions with a discrete stable law, which have power-law tails and whose moments and autocorrelation function do not exist. The separate problem of monitoring and characterizing the fluctuations is studied, analysing the statistics of individuals that leave the population. The fluctuations in the size of the population are transferred to the times between emigrants that form an intermittent time series of events. The emigrants are counted with a detector of finite dynamic range and response time. This is modelled through clipping the time series or saturating it at an arbitrary but finite level, whereupon its moments and correlation properties become finite. Distributions for the time to the first counted event and for the time between events exhibit power-law regimes that are characteristic of the fluctuations in population size. The processes provide analytical models with which properties of complex discrete random phenomena can be explored, and in addition provide generic means by which random time series encompassing a wide range of intermittent and other discrete random behaviour may be generated.

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1. Introduction

The richness in behaviour exhibited by complex systems and networks stems from their being far from equilibrium and in a highly correlated state. The fluctuations in these systems are large and frequently described by scale-free probability density functions. Recently, manifestations of scale-free fluctuations have been observed in discrete random phenomena in contrast to more familiar examples that describe continuous behaviour. One example of this is the order distribution of networks, being a measure of the complexity exhibited by diverse systems such as the Internet and WWW [1, 2], organic metabolisms [3], protein interactions [4] and social networks [5]. The order distribution describes the number N of incoming or outgoing links that connect nodes within the system, and it is found that these are typically scale free with discrete probability distribution function having a power-law tail, $p(N) \sim 1/N^{\gamma}$. This contrasts with a 'random network' where the order distribution is Poisson [6]. Another example is presented by the now classical paradigm of a complex system, the sandpile. The distance travelled by grains in an avalanche, the 'flight length', is a continuous random variable with power-law density. This is paradoxical, for the variance of the distance travelled does not exist, and so the energy required to transport the particles is apparently infinite. In fact, a more careful analysis reveals [7, 8] that the particles' flight lengths comprise a sum of steps where the length of each step is a random variable with finite variance. However, the *number* of steps N is a discrete random variable with probability distribution having a power-law tail, so that once again $p(N) \sim 1/N^{\gamma}$. The power-law behaviour of the flight-length distribution is therefore inherited from the number of times a grain is perturbed during an avalanche.

Situations in which such discrete probability distributions can arise is the first topic to be treated in this paper. Models for the order parameter in complex networks have been derived, but these analyses are confined to those instances where N is large and can be approximated by a continuum, the so-called mean-field approximation. Here we recognize the fact that N is a discrete variable and develop two simple stochastic processes that can produce distributions with power-law tails as special cases of their more general behaviour. These processes are variations upon birth–death–immigration (BDI) processes that have been widely studied in the mathematical literature [9] and have featured, for example, in optics applications as models for the laser below threshold [10] and for clutter that limit the performance of coherent detection systems [11]. A particular virtue of these models is that in addition to providing an analytical means for generating single-fold probability distributions with power-law characteristics, the joint properties are also completely defined by the process, enabling correlations to be studied within the same theoretical framework.

In reviewing population models, the simplest case to consider in the first instance is the birth–death process. Births and deaths occur in proportion to the instantaneous size of the population and cause fluctuations. However, such a process is unstable, having a stationary population size that is zero or infinite unless the birth and death rates are identical. This process can be stabilized by the ingress of immigrants to the population, provided the death rate exceeds the birth rate. The immigrants in most models enter singly and at a constant rate, instantaneously increasing the size of the population by 1. In this case, the stationary solution for the process is the negative binomial class of probability distributions, one member of which is the geometric or Bose–Einstein distribution that describes the fluctuations of thermal light. Other models exist that extend this process to when immigrants enter the population in pairs [12], the motivation being the study of squeezed states of non-classical light. The steady state of this process has a distribution involving Laguerre polynomials and these exhibit odd–even effects, illustrating the non-trivial effect that immigration of multiples can have.

A variation of the BDI process to one with no births but with immigrations occurring singly, in pairs, ..., m-tuplets was introduced [13] to study how models having identical single-fold probability distributions can nevertheless be differentiated through a study of the higher order joint statistical properties. This work was motivated by the need to develop generic tools for distinguishing between candidate models of a complex physical process. The death–multiple immigration (DMI) model where m-tuplet immigrants arrive with rates α_m that are particular to their order facilitates this characterization problem because of a useful property of the stationary state of the process. The stationary state can be tailored to have a pre-selected form by uniquely prescribing the rates at which the m-tuplet immigrants arrive. The DMI model can be generalized to a birth–death–multiple immigration (BDMI) model, which is a different stochastic process that nevertheless can also be tailored to have a pre-selected stationary solution. It is this property that we will exploit in the first part of the present paper, where we determine the rates for the immigrants that produce a stationary state of the population that belong to the class of discrete stable distributions [14].

Systems described by power-law distributions or densities are typified by large fluctuations and this prompts the practical question of how the behaviour of such populations can be measured, the treatment for which constitutes the second part of the paper. The methodology used [15] for counting photoelectrons is particularly appropriate for this purpose. This technique obtains the joint probability distribution for there being N members present in the stationary population, and with n emigrants having been counted in a specified time interval, usually called the integration time of the counting mechanism. The emigrations are treated as additional deaths within the population and constitutes a series of emissions that is modulated by fluctuations in this population. In this way, the fluctuations in the internal population size are transferred to an external series of countable events.

Although the counting distribution is not of the discrete stable class, it does retain the power-law character of the monitored population, and so its moments and correlation functions are not defined. This prompts the final question that the paper addresses, which is how the fluctuating time series can be characterized using familiar statistical measures. To address this aspect of the problem we recognize and exploit the fact that any detection scheme has a finite dynamic range that will limit or saturate the number of counted emissions, and a finite response time so that the detector is incapable of resolving events that occur in too rapid a succession. The paper concludes with a discussion of the broader implications of this work, and the avenues by which it can be developed.

2. Multiple immigration models

This section considers two population models, one describing deaths and multiple immigrations, the other births, deaths and multiple immigrations. The general solution of these two processes will be derived and then particular solutions that have a discrete stable law as the stationary state of the population will be obtained.

2.1. The death–multiple immigration (DMI) process

The population evolves according to a death–multiple immigration model (DMI). The population size increases through immigration of singles, pairs, . . . , m-tuplets, . . . , which arrive at rates $\alpha_m \geqslant 0$, and is depleted at a constant rate μ by deaths that occur singly in proportion to the instantaneous size of the population. The rate equation for this process

$$\frac{\mathrm{d}P_N(t)}{\mathrm{d}t} = \mu(N+1)P_{N+1} - \mu N P_N - P_N \sum_{m=1}^{\infty} \alpha_m + \sum_{m=1}^{N} \alpha_m P_{N-m} \tag{1}$$

describes the evolution of $P_N(t)$, the probability that the population comprises N members at time t. The solution of equation (1) can be found using the generating function

$$Q(s;t) = \langle (1-s)^N \rangle = \sum_{N=0}^{\infty} (1-s)^N P_N(t) \tag{2}$$

from which factorial moments and probabilities can be determined:

$$\frac{\langle N(N-1)(N-2)\cdots(N-r+1)\rangle}{\langle N\rangle^r} = \left(-\frac{1}{\langle N\rangle}\frac{\mathrm{d}}{\mathrm{d}s}\right)^r Q(s;t)\bigg|_{s=0}$$
$$P_N(t) = \frac{1}{N!}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\right)^N Q(s;t)|_{s=1}. \tag{3}$$

The single-fold generating function satisfies the partial differential equation,

$$\frac{\partial Q}{\partial t} = -\mu s\frac{\partial Q}{\partial s} - F(s)Q \tag{4}$$

where

$$F(s) = \sum_{m=1}^{\infty}\alpha_m(1-(1-s)^m) \tag{5}$$

and must be solved subject to the boundary conditions $Q(0,t) = 1$ and $Q(s;0) = Q_0(s) = (1-s)^M$ which imply that the probability distribution has unit normalization at all times and that the population initially has M members present.

The solution of equation (4) can be written as [16]

$$Q(s;t) = Q_0(\Phi(s;t))\exp\left(-\int^t \mathrm{d}t'\Phi(s,t')\right)$$

where

$$\frac{\partial\Phi}{\partial t} + \mu s\frac{\partial\Phi}{\partial s} = 0$$

and

$$\Phi(s;0) = F(s).$$

The stationary solution for the generating function is

$$Q(s;\infty) = Q_{\mathrm{st}}(s) = \exp\left(-\int^s \mathrm{d}s'\frac{F(s')}{\mu s'}\right) \tag{6}$$

which is evidently dependent upon the rates α_m at which the immigrants arrive. Moreover, by judicious selection of the function $F(s)$ it is possible to tailor the process to have a specific stationary state. The condition that the DMI process can produce a pre-selected stationary solution is contingent upon the convergence of the series given by equation (5) and that the immigration rates $\alpha_m \geqslant 0$ for all $m \geqslant 1$. There is a subtle and non-trivial relationship between the immigration rates and the stationary solution. For example, if the rates are selected from one member of the negative binomial class of distributions, e.g. the geometrical distribution with $\alpha_m = a\xi^m$ where $a > 0$ and $0 < \xi < 1$, then

$$F(s) = \frac{a\xi s}{(1-\xi)(1-\xi(1-s))}$$

and so

$$Q_{\mathrm{st}}(s) = \left(1 + \frac{\xi}{1-\xi}s\right)^{-a/(1-\xi)}$$

which is the generating function for the *entire* negative binomial class of distributions [9].

The continuous Lévy stable distributions [17] are defined through their characteristic function, this being the Fourier transform of the probability density function. These distributions are invariant under convolution with themselves, hence the epithet 'stable', and have probability distribution functions with power-law tails $p(x) \sim 1/x^{1+\upsilon}$ where $0 < \upsilon < 2$,

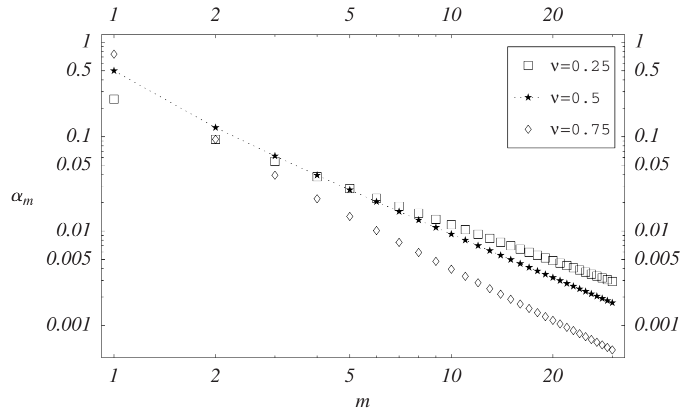

Figure 1. Immigration rates α_m, for the DMI process, as a function of the order of the m-tuplet for $\nu = 1/4$, $1/2$ and $3/4$ [14].

so the *variance* does not exist. The special case $\upsilon = 2$ is the familiar Gaussian distribution. By analogy, it is possible to define a class of discrete stable distributions through their generating function

$$Q(s) = \exp(-As^{\nu}). \tag{7}$$

This defines positive definite distributions provided that $0 < \nu \leqslant 1$, and where A is a positive, real constant that acts as a (unimportant) scaling factor. These discrete variables N have probability distribution functions with power-law tails for $0 < \nu < 1$, namely $P_N \sim 1/N^{1+\nu}$, so that the *mean* does not exist. The special case $\nu = 1$ when inserted in equation (7) gives the familiar Poisson distribution, which does not have a power-law asymptote. The Poisson distribution therefore has an analogous status to discrete stable distributions as the Gaussian does to continuous stable densities. We shall now show how the class of discrete stable distributions form the stationary state of a particular DMI process.

From equations (6) and (7) the unique choice of $F(s)$ that gives the stable law generating function is

$$F(s) = A\mu\nu s^{\nu} \equiv as^{\nu}.$$

Using equation (5) the immigration rates can be determined through writing

$$as^{\nu} \equiv a(1 - (1 - s))^{\nu} = \sum_{m=1}^{\infty} \alpha_m (1 - (1 - s)^m)$$

the left-hand side of which can be expanded in powers of $(1 - s)$ because $0 \leqslant s \leqslant 1$, whereupon the coefficients α_m are determined to be

$$\alpha_m = -\frac{a\Gamma(m - \nu)}{\Gamma(-\nu)m!}. \tag{8}$$

These coefficients are all positive only if $0 < \nu < 1$, which covers the whole parameter range for the power-law regime of the discrete stable distributions. Moreover the rates are independent of the death rate, this being the other parameter in the model. The special case $\nu = 1$ corresponds to no multiple immigrations, i.e. $\alpha_1 = a$ and $\alpha_m = 0$ for $m \geqslant 2$, which is the Poisson process. Figure 1 shows α_m as a function of the order of the multiplets for $a = 1$, $\mu = 2$ and $\nu = 1/4$, $1/2$ and $3/4$, and reveals that the immigration rates themselves have power-law tails for large values of m, i.e. $\alpha_m \sim 1/m^{1+\nu}$. This illustrates the unique

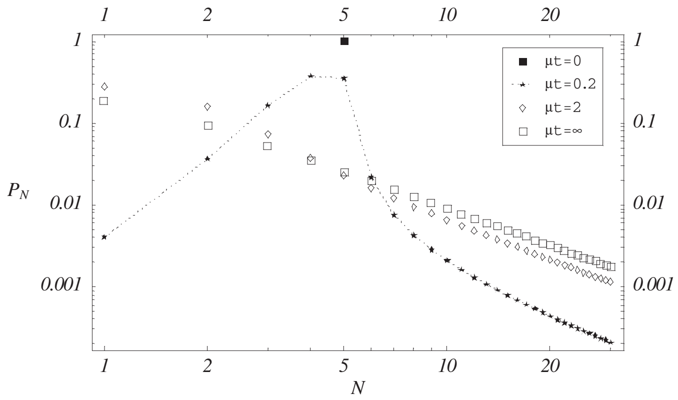

Figure 2. Temporal evolution of the PDF of the death–multiple immigration population model initiated with $M = 5$, $\nu = 1/2$, $a = 1$ and the death rate $\mu = 2$ for times $\mu t = 0$ (■), 0.2 (★), 2 (◇) and ∞(□). The $N^{-3/2}$ tail of the distribution is established immediately [14].

dominance of the power-law behaviour in contrast to the case when immigrant arivals are governed by a geometrical distribution, for example, which we have seen generates an entire distribution class from one member of that class. The $\mu s'$ term in the denominator of the integrand appearing in equation (6) is a consequence of deaths in the underlying process. Its presence means that for power-laws, and only for power-laws, the immigration rates are mirrored by the stationary distribution of the population itself.

With the choice of immigration rates given by equation (8), the PDE for the generating function is

$$\frac{\partial Q}{\partial t} = -\mu s \frac{\partial Q}{\partial s} - a s^{\nu} Q$$

whose solution can be found by elementary means to be

$$Q^{[M]}(s;t) = Q_0(f(s;t), M) Q_E(s;t) = (1 - f(s;t))^M \exp\left[-\frac{a s^{\nu}}{\nu\mu}(1 - \exp(-\nu\mu t))\right] \tag{9}$$
$$f(s;t) = s\exp(-\mu t).$$

This describes the evolution from an initial state $P_N(0) = \delta_{N,M}$ to a stationary state with stable law generating function:

$$Q_E(s;\infty) = Q_{\mathrm{st}}(s) = \exp(-a s^{\nu}/\nu\mu). \tag{10}$$

The evolution of the distribution corresponding to equation (9) is illustrated in figure 2 for the case when $\nu = 1/2$, this being one of the few cases for which the stationary distribution can be expressed in closed form, namely,

$$P_N = \frac{2}{\pi^{1/2} N!}\left(\frac{a}{\mu}\right)^{N+1/2} K_{N-1/2}\left(\frac{2a}{\mu}\right)$$

with $K_\nu(z)$ a modified Bessel function [18]. This distribution has power-law tail with $P_N \sim N^{-3/2}$, as can be seen in figure 2 where the other parameters are $a = 1$, $\mu = 2$ and $M = 5$. Note that the tail of the distribution is established immediately, implying that moments of the distribution do not exist for any $t > 0$. This is because the rates α_m permit a large number of immigrants to enter at a rate that is not exponentially bounded.

The joint generating function that describes the population having sizes N and N' following a delay time t, can be deduced from the stationary solution together with equation (9), conditioned upon there being N members initially present, namely,

$$
\begin{aligned}
Q(s, s'; t) = \langle (1-s)^N (1-s')^{N'} \rangle &= \sum_{N,N'=0}^{\infty} (1-s)^N (1-s')^{N'} P_N P(N'|N) \\
&= \sum_{N=0}^{\infty} Q^{[N]}(s'; t) P_N (1-s)^N \\
&= \exp\left[-\frac{a}{\nu\mu} (s'^{\nu}(1-\exp(-\nu\mu t)) + (s + (1-s)s' \exp(-\mu t))^{\nu}) \right] \qquad (11)
\end{aligned}
$$

from which joint distributions and, in principle, autocorrelation and higher order statistical measures can be obtained. However, because the joint probabilities also have power-law tails, the autocorrelation function is not defined. A noteworthy property of equation (11) is that it is not invariant to the interchange of s with s', which implies that the death–multiple immigration population model does not possess a doubly stochastic representation [13]. Hence the population cannot be regarded as evolving in response to a continuous random fluctuation: in quantum optics terms the process is non-classical. Equations (9) and (11) provide a closed form solution for the single-fold and joint evolution of a stochastic process with discrete power-law stationary distribution. The next section shows how a similar stationary state can be obtained from a different stochastic process that incorporates births into the formulation.

2.2. The birth–death–multiple immigration (BDMI) process

The population in this case evolves in a similar fashion to that of the process detailed above. However, in this instance it not only increases due to multiple immigrants but also due to births that occur at a constant rate λ and in proportion to the instantaneous size of the population. The rate equation for the BDMI process is

$$
\frac{\mathrm{d}P_N(t)}{\mathrm{d}t} = \mu(N+1)P_{N+1} - (\lambda+\mu)NP_N + \lambda(N-1)P_{N-1} - P_N \sum_{m=1}^{\infty} \alpha_m + \sum_{m=1}^{N} \alpha_m P_{N-m} \qquad (12)
$$

and the corresponding generating function is a solution of

$$
\frac{\partial Q}{\partial t} = -s(\mu - \lambda(1-s)) \frac{\partial Q}{\partial s} - F(s)Q \qquad (13)
$$

where $F(s)$ is given by equation (5). The general solution is given by

$$
Q(s; t) = Q_0(\Psi(s; t)) \exp\left(-\int^t \mathrm{d}t' \, \Psi(s, t') \right)
$$

where

$$
\frac{\partial \Psi}{\partial t} + s(\mu - \lambda(1-s)) \frac{\partial \Psi}{\partial s} = 0
$$

and

$$
\Psi(s; 0) = F(s).
$$

A stationary solution of this process only exists if the death rate exceeds the birth rate, $\mu > \lambda$, in which case

$$
Q(s; \infty) = Q_{\mathrm{st}}(s) = \exp\left(-\frac{1}{\mu - \lambda} \int^s \mathrm{d}s' \frac{F(s')}{s'(1+bs')} \right)
$$

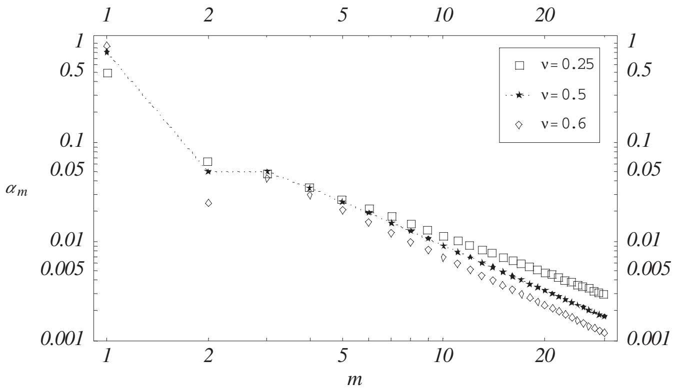

Figure 3. Immigration rates α_m, for the BDMI process, as a function of the order of the m-tuplet for $\nu = 1/4$, $1/2$ and $3/5$.

where $b = \lambda/(\mu - \lambda) > 0$. It can be seen that a prescribed stationary solution of this process can be obtained, but that the function $F(s)$ will necessarily have a different form to the DMI process, for which $b = 0$. This in turn means that the immigration rates α_m will be dependent on the birth and death rates.

For the BDMI process to have a stable law generating function as its stationary state requires $F(s)$ to be given by

$$F(s) = A(\mu - \lambda)\nu(1 + bs)s^{\nu} \equiv a(1 + bs)s^{\nu}.$$

Upon expanding this in powers of $(1 - s)$ and equating the result to terms in the expansion given by equation (5) gives the immigration rates:

$$\alpha_m = \frac{a\Gamma(m - \nu)(m\lambda - (m - 1 - \nu)\mu)}{\Gamma(-\nu)m!(\mu - \lambda)(m - 1 - \nu)}. \tag{14}$$

Figure 3 shows the form of the rates for $a = 1$, $\mu = 12/5$, $\lambda = 2/5$ and $\nu = 1/4$, $1/2$ and $3/5$. Note than when $\lambda = 0$ equation (14) reduces to equation (8). It is interesting to note that for the BDMI process, the α_m are dependent on the birth and death rates in addition to ν whereas the DMI process contains the constant ν alone. The rates (14) are all positive provided that $0 < \nu < 1 - 2\lambda/\mu$, which necessarily reduces the range of parameters for which this process has the stationary solution of choice. The generating function satisfies the PDE

$$\frac{\partial Q}{\partial t} = s(-\mu + \lambda(1 - s))\frac{\partial Q}{\partial s} - a\nu(1 + bs)s^{\nu}Q \tag{15}$$

with solution

$$Q(s; t) = Q_0\left(\frac{s\theta}{1 + bs(1 - \theta)}\right)\exp\left[\frac{-as^{\nu}}{(\mu - \lambda)\nu}\left(1 - \frac{\exp(-\nu(\mu - \lambda)t)}{(1 + bs(1 - \theta))^{\nu}}\right)\right]$$
$$\theta = \exp(-(\mu - \lambda)t). \tag{16}$$

The stationary solution takes the form

$$Q(s; \infty) = Q_{\mathrm{st}}(s) = \exp(-as^{\nu}/\nu(\mu - \lambda)) \tag{17}$$

which has an identical form to the stationary solution of the DMI process, apart from a different parameterization. The distribution has power-law tail $P_N \sim N^{-(1+\nu)}$ with index in the range -1 to $-2(1 - \lambda/\mu)$. This suggests that for a particular choice of parameters, the stationary single-fold distributions of both processes can be identical and any difference

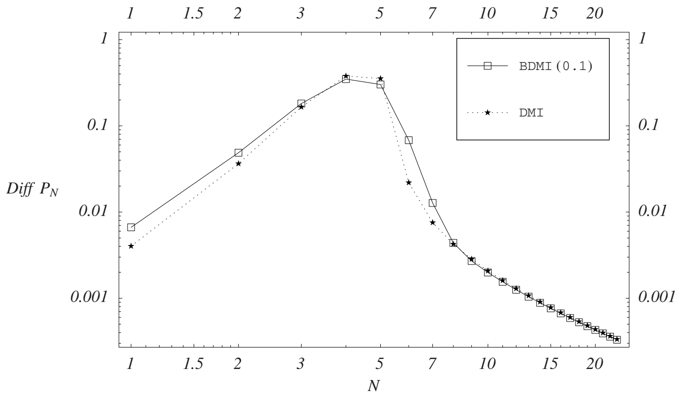

Figure 4. Comparison of the temporal evolution of the BDMI (□) and DMI (★) population models, initiated with $M = 5$, $\nu = 1/2$, $a = 1$ for both processes and $\mu = 2$ for the DMI process and $\mu = 12/5$, $\lambda = 2/5$ for the BDMI process. A difference in the PDFs is apparent until the time-dependent behaviour dies away.

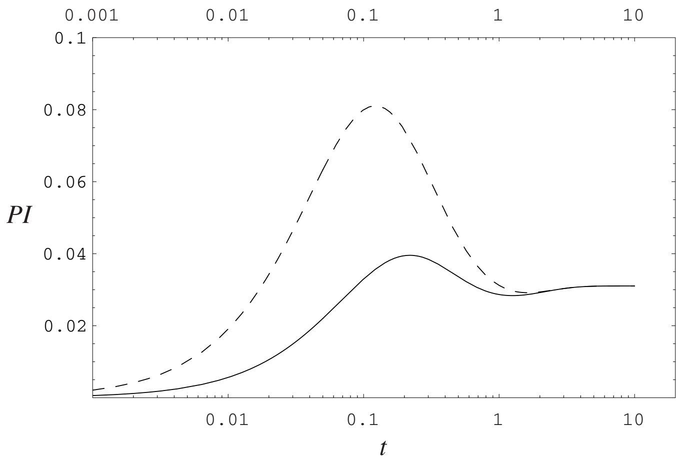

Figure 5. Plot of the probability of a population increase (PI) as a function of t, where the full and dashed lines represent the DMI and BDMI process respectively. This represents the probability $p(N = 4, t|N = 3, t + \Delta t)$, where in this instance $N = 3$.

will appear in their transient and correlation properties or from properties derived from these. Figure 4 shows a plot of the transient solutions for the DMI and BDMI processes, for $t = 0.1$. The choice of parameters leads to the same stationary state as that shown in figure 2, but the transient behaviour differs, albeit by a small amount. This difference can be discerned by consideration of the time for the population size to change by a similar value for either process. Because the BDMI includes births, the probability for the population increasing from N to $N + 1$ in a time interval Δt will be greater than that for the DMI process. Figure 5 shows an example of this, the probability of a population increase (PI) by 1 in a time Δt

$$\mathrm{PI} = p(N = 4, t + \Delta t|N = 3, t).$$

We can calculate the joint generating function that takes the form

$$Q(s, s'; t) = \exp\left[-\frac{a}{\nu(\mu-\lambda)}\left(s'^{\nu}\left(1-\frac{\exp(-\nu(\mu-\lambda)t)}{(1+bs'(1-\theta))^{\nu}}\right)+\left(s+\frac{(1-s)s'\theta}{1+bs'(1-\theta)}\right)^{\nu}\right)\right]. \tag{18}$$

Once again the autocorrelations and higher order statistical measures are undefined because the joint probabilities have power-law tails. Note that if $\lambda = 0$, the births are removed and equation (18) reduces to the joint generating function of the DMI process.

3. External monitoring of a population process

It is often impossible to make a direct measurement of the population but rather only some external manifestation of its evolution. Many experimental situations externally monitor the evolution by counting the number n of emigrants that leave the population at rate η. These emigrants form a series of events that can be counted in time intervals of duration T, the integration time of the detector [12]. Formulating the externally monitored counting process necessitates introducing the joint probability distribution $p_{N,n}(T)$ for a stationary population of size N and with n emigrants having been counted in the integration time interval [0, T]. Modelling the counted emigrants requires the inclusion of an additional death process to equation (1). This produces the following rate equation for the DMI process where the last two terms relate to the deaths due to emigrations [16]:

$$\frac{\mathrm{d}p_{N,n}(T)}{\mathrm{d}T} = \mu(N+1)p_{N+1,n} - \mu N p_{N,n} - p_{N,n}\sum_{m=1}^{\infty}\alpha_m + \sum_{m=1}^{N}\alpha_m p_{N-m,n} + \eta(N+1)p_{N+1,n-1} - \eta N p_{N,n}. \tag{19}$$

This can be solved with the aid of the joint generating function for the counted population

$$Q_c(s, z; T) = \langle (1-s)^N (1-z)^n \rangle. \tag{20}$$

When the immigration rates are selected according to equation (8), the generating function for the integrated statistics is a solution of

$$\frac{\partial Q_c}{\partial T} = (\eta z - \tilde{\mu} s)\frac{\partial Q_c}{\partial s} - as^{\nu} Q_c \tag{21}$$

where $\tilde{\mu} = \mu + \eta$ is a composite death rate, and the solution is initiated from the stationary solution for the population, so that $Q_c(s, z; 0) = Q_{\mathrm{st}}(s)$. Hence the generating function for the integrated statistics is

$$Q_c(s, z; T) = Q_{\mathrm{st}}(\Phi)Q_1(s, z; T) = Q_{\mathrm{st}}(\Phi)\exp\left(-\frac{a}{(1+\nu)\eta z}(\Phi(s, z; T)^{\nu+1}F(1+\nu, 1, 2+\nu; \tilde{\mu}\Phi(s, z; T)/\eta z) - s^{\nu+1}F(1+\nu, 1, 2+\nu; \tilde{\mu}s/\eta z))\right) \tag{22}$$

$$\Phi(s, z; T) = [\eta z + (\tilde{\mu}s - \eta z)\exp(-\tilde{\mu}T)]/\tilde{\mu}$$

where $F(a, b, c; x)$ is the hypergeometric function [18]. This methodology can also be applied to the BDMI process, which leads to the generating function for the integrated statistics being

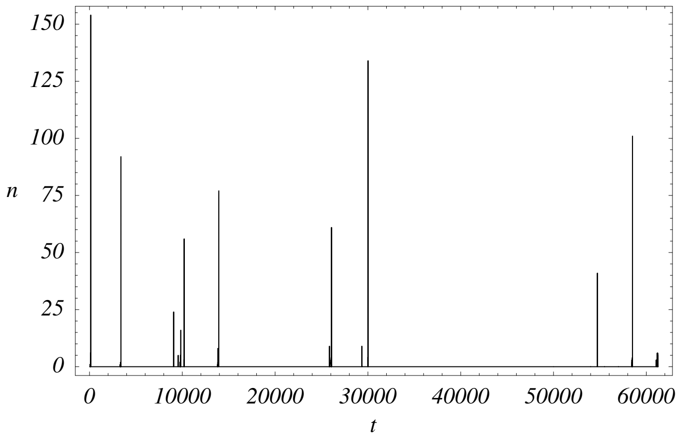

Figure 6. A realization of the counted series of events, with $\tilde{\mu} = 3$ and the other parameters identical to those in figure 1. The integration time $T = 5$. The emanations show wide variability in the size of events and also in the times of their occurrence [14].

$$
\begin{aligned}
Q_c(s, z; T) &= \exp(-(G(s) - G(\Psi))) \exp\left(-\frac{a\Psi^{\nu}}{(\tilde{\mu} - \lambda)\nu}\right) \\
G(s) &= \frac{a}{\lambda(u_+ - u_-)}\left[\frac{s^{\nu+1}}{\nu+1}\left(\frac{1}{u_-}F(1, 1+\nu, 2+\nu; s/u_-) - \frac{1}{u_+}F(1, 1+\nu, 2+\nu; s/u_+)\right.\right. \\
&\quad \left.\left. + \frac{\tilde{b}(1+\nu)s}{(2+\nu)}\left(\frac{1}{u_-}F(1, 2+\nu, 3+\nu; s/u_-) - \frac{1}{u_+}F(1, 2+\nu, 3+\nu; s/u_+)\right)\right)\right] \\
\Psi(s, z; T) &= \frac{u_+(s - u_-)\exp(\lambda u_+ T) - u_-(s - u_+)\exp(\lambda u_- T)}{(s - u_-)\exp(\lambda u_+ T) - (s - u_+)\exp(\lambda u_- T)} \\
u_{\pm} &= \frac{\pm(1 + 4\tilde{b}\tilde{c}z)^{1/2} - 1}{2\tilde{b}} \qquad \tilde{b} = \frac{\lambda}{\tilde{\mu} - \lambda} \qquad \tilde{c} = \frac{\eta}{\tilde{\mu} - \lambda}.
\end{aligned}
\tag{23}
$$

Denoting $Q_c(z; T) = Q_c(0, z; T)$ in conjunction with equation (3) and differentiating with respect to the z variable formally obtains factorial moments and distributions for the counted series of events comprising emigrants alone. The counting distribution for both processes has power-law tails for all integration times T, an observation that distinguishes this integrated process from those with finite moments, which necessarily become Poissonian in the large T limit. This is because the events within a long integration time can be decomposed into many independent groups, each of which is approximately stably distributed, and thereby so is their sum.

Using the methods described in [19] it is possible to simulate the behaviour of these models and thereby produce time series for the processes. A realization of the DMI process is illustrated in figure 6. Note the variation of size and the intermittent nature of the emissions, both being manifestations of the power-law distribution.

These discrete stochastic population processes have stationary solutions possessing power-law characteristics in the stable regime. The emigrations from these populations form a series of discrete events that can be sampled. The resulting time series is a realization of the behaviour of the parent population, also possessing power-law characteristics thereby demonstrating the dominance of the embedded power-law. The nature of the power-law prevents the formation of any standard statistics other than classification using the probability distributions of fluctuations in the time series. The next section considers the limitations that any real detection system has upon measuring these fluctuations, and how this can be incorporated advantageously to regularize the fluctuations.

4. Construction of statistical measures by clipping

The characterization of any real population process or series of events will be limited in practice by the dynamic range and time resolution of the measuring or 'detection' system as well as by the finite time available for the measurement. These limitations are of particular concern in the present context for several reasons. In particular, the probability of finding a very large fluctuation in the number of individuals leaving the population is exceptionally high for power-law processes and the detector may saturate so that these events do not register correctly. Moreover the chance of observing the most extreme events in an experiment of finite duration is small. Hence the tail of their distribution will be ill-defined and statistical measures such as moments and correlation functions will always be finite but will change with measurement time. Thus the registered train of events will generally be different from that predicted by the ideal population model discussed earlier and although it may retain some characteristics of the original series, the loss of information during detection may lead to modified statistical properties and reduced measurement accuracy. One method by which this problem can be controlled and quantified is to employ the technique of 'clipping' or 'limiting'.

Clipping is a technique more familiar in the context of analogue signal processing and was originally developed as a method for 'jamming' radar and communication systems [20]. The term is generally applied to a process in which the original signal is replaced by a telegraph wave that takes values ± 1 according as the signal lies above or below a chosen level. In the case of a Gaussian signal clipped at zero or *hard limited*, the spectrum is broadened according to the well-known 'arcsine' formula or Van Vleck theorem [21]. A modified version of the technique was first applied to digital signals in the late 1960s as a means of simplifying the post detection processing of series of photoelectric events in light scattering experiments [22]. The quantity of interest was the autocorrelation function of the series and this required (at the time) unattainably rapid multiplication of the numbers of events in samples recorded at different times. To overcome this problem the number of counts registered in each sample time was replaced by 1 or 0 as follows:

$$c_k(t, T) = \begin{cases} 0 & 0 \leqslant n(t, T) \leqslant k \\ 1 & n(t, T) > k. \end{cases} \tag{24}$$

It can be shown that the shape of the autocorrelation function of detected thermal light is not changed by clipping *one* autocorrelation channel in this way: a procedure that greatly simplified the multiplication process at a small cost in statistical accuracy [23]. Various refinements of the method, such as the use of a distribution of clipping levels and the technique of *scaling*, were developed to avoid distorting the correlation function in the case of other kinds of light [24].

In view of the large excursions in count rate predicted for power-law population models, the procedure (24) might well be necessary in the present context to avoid the unknown non-linear behaviour of any counting device. Alternatively, it could be adopted as a simple idealized model for effects of this kind. Of course the application of such a procedure will change both the statistics and correlation properties of the stream of individuals leaving the population. The distribution of the new counting process will evidently be different and, more importantly for our purposes, its moments and correlation functions will be finite. The question is: is it possible to make realistic measurements that are sensitive to the parameters of the model? In order to begin to answer this question it is useful to consider the simplest but most extreme form of (24) when the series of events is hard limited:

$$c_0(t,T) = \begin{cases} 0 & n(t,T) = 0 \\ 1 & n(t,T) > 0. \end{cases} \tag{25}$$

Since the data now consist purely of 0 and 1 the generating function of the clipped distribution can be written as

$$Q_{\mathrm{cl}}(z,T) = \sum_{n=0}^{1} (1-z)^n p_n = p_0(T) + (1-z)(1-p_0(T)). \tag{26}$$

Using equation (3) and $p_0(T) = Q_c(1,T)$, the clipped mean takes the form

$$\bar{c}(T) = 1 - p_0(T) = 1 - Q_c(1,T) \tag{27}$$

which depends on integration time. This process may seem a little severe, however, we can make the comparison between the above process and that of saturating the signal at 2 counts. This saturated signal takes its true values for registered counts of 0 and 1 and then registers 2 for all subsequent counts. The generating function for this scenario is

$$Q_{\mathrm{sat}}(z,T) = \sum_{n=0}^{2} (1-z)^n p_n = p_0(T) + (1-z)p_1(T) + (1-z)^2(1 - p_0(T) - p_1(T)) \tag{28}$$

where $p(1,T)$ is defined by equation (3). The saturated mean is

$$\begin{aligned} \frac{\bar{c}_{\mathrm{sat}}(T)}{2} &= 1 - p_0(T) - \frac{1}{2}p_1(T) \\ &= 1 - Q_c(1,T) + \frac{1}{2}\left.\frac{\partial Q_c(z,T)}{\partial z}\right|_{z=1}. \end{aligned} \tag{29}$$

The probability $p_r(T)$ of counting r emigrants can be calculated from Q_c. For the DMI process, when $\tilde{\mu}T \gg 1$, the generating function (22) takes the form

$$Q_c(1,T) = \exp\left(\frac{-a}{\tilde{\mu}}\left(\frac{\eta}{\tilde{\mu}}\right)^{\nu} \frac{\nu}{1+\nu}\tilde{\mu}T\right) \tag{30}$$

which in the limit $T \to \infty$ is zero and so $\bar{c}(T)$ and $\bar{c}_{\mathrm{sat}}(T)$ are asymptotically unity and 2 respectively. This can be seen in figure 7, which shows $\bar{c}(T)$ and $\bar{c}_{\mathrm{sat}}(T)$, normalized by unity and 2 respectively for the DMI process. The BDMI process has similar behaviour. The figure shows that in the limit $\tilde{\mu}T \ll 1$, the clipped mean for both curves does not increase linearly but rather has power-law dependence

$$\bar{c}(T) \sim \frac{a}{\nu\tilde{\mu}}(\eta T)^{\nu} + O(T^{2\nu}). \tag{31}$$

This indicates that even the most severely limited integrated counting measurements retain a remnant of the scale-free behaviour of the monitored population and no extra information is gained by increasing the saturation limit. Hence from this point all results will pertain to the process of hard limiting.

It is also possible to calculate the autocorrelation function for the hard limited process. This can be found using the joint distribution of counts in two non-overlapping integration periods of length T, separated by an interval Δt. The derivation is quite lengthy, has been presented elsewhere and so details that are particular to the present application are consigned to the appendix. The autocorrelation function for the hard limited case is

$$\zeta(T,\Delta t) = \frac{\langle c_0(0,T)c_0(T+\Delta t,T)\rangle}{\langle c_0(T)\rangle^2} = \frac{(1 - 2Q_{\mathrm{cl}}(1;T) + R(1,1;T+\Delta t,T))}{(1 - Q_{\mathrm{cl}}(1;T))^2}. \tag{32}$$

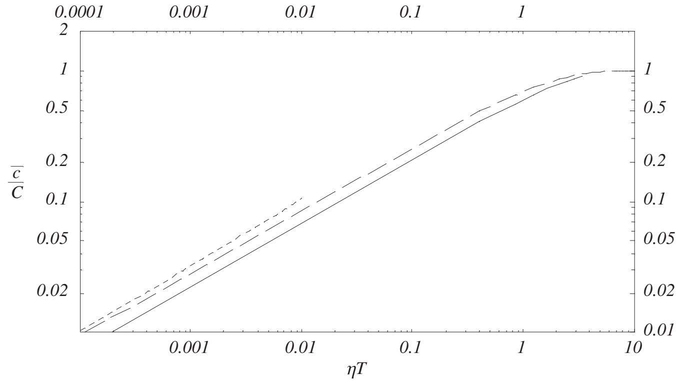

Figure 7. Illustrating the dependence of the hard limited mean and saturated mean, represented by the full and large dashed line respectively, on the integration time T. The small dashed line shows the power-law asymptote. The parameters used are the same as those in figure 4. C represents that both curves have been normalized by 1 and 2 respectively.

The function R is the generating function for the joint distribution of counts in two non-overlapping equal time intervals, which is defined at the end of the appendix. Due to similar general behaviour of the autocorrelation for both processes, it is only necessary to plot the results of one of the processes. Figure 8(*a*) shows the dependence of the autocorrelation on the separation time, Δt; between successive samples for three values of integration time T for the DMI process. For increasing separation time the measurements de-correlate. Figure 8(*b*) displays the autocorrelation for the DMI process as a function of integration time T. As can be seen the autocorrelation possesses power-law divergence for small integration times, indicating the divergence of the moments. Note that the autocorrelation tends to unity for large integration times, a standard characteristic of Markovian stochastic processes. For $\eta T \ll 1$ and in the limit $\Delta t \to 0$, the autocorrelation for the DMI process is

$$\zeta_{\mathrm{DMI}}(T,0) = \frac{\tilde{\mu}\nu}{a}(2-2^{\nu})(\eta T)^{-\nu} \tag{33}$$

whose divergence for small T once again indicates the non-existence of the moments for the monitored series of events. The corresponding behaviour for the BDMI process is

$$\zeta_{\mathrm{BDMI}}(T,0) = \frac{(\tilde{\mu}-\lambda)\nu}{a}(2-(1+\xi)^{\nu})(\eta T)^{-\nu} \qquad \xi = (1+4\tilde{b}\tilde{c})^{1/2} \tag{34}$$

which has a similar dependence on the integration time as the DMI process (33). The ratio of these two leading terms in the autocorrelation functions is

$$\frac{\zeta_{\mathrm{DMI}}}{\zeta_{\mathrm{BDMI}}} = \frac{1-2^{\nu-1}}{1-\frac{(1+\xi)^{\nu}}{2}} > 1 \tag{35}$$

for a choice of constant that produces identical stationary behaviour and is therefore a distinguishing feature of the two models. This should be contrasted with the case considered previously [13] which advocated using higher order correlation properties to distinguish two processes.

Other measurements relating specifically to the time series of events can be calculated. The probability density for the time τ to the first count, $w_0(\tau)$, can be defined as follows [15]:

$$w_0(\tau) = -\frac{\partial Q_{\mathrm{cl}}(1;\tau)}{\partial \tau} = -\frac{\partial Q_c(1;\tau)}{\partial \tau}. \tag{36}$$

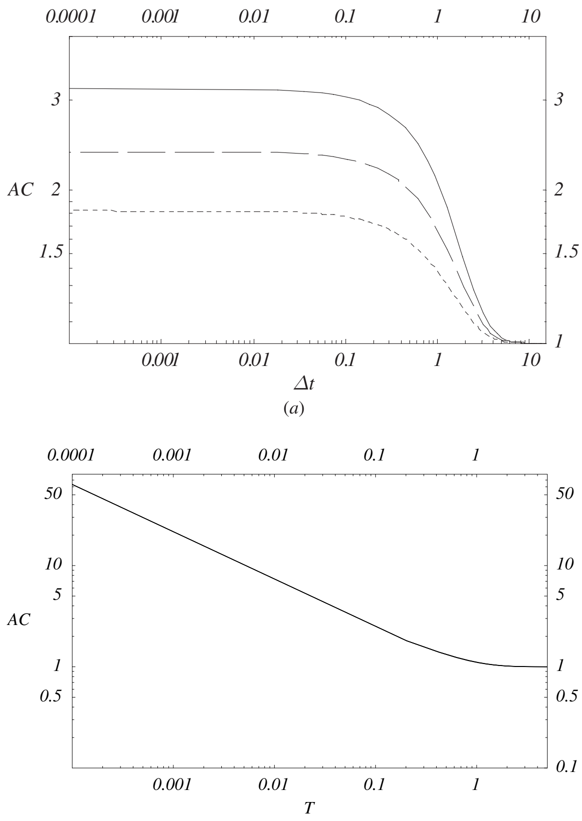

Figure 8. (*a*) The curves show the dependence of the autocorrelation on separation time of successive samples for $T = 0.05$ (solid line), 0.1 (large dashed line), 0.2 (small dashed line). (*b*) Illustrating the dependence of the autocorrelation, for the DMI process, on the integration time *T*. The parameters used for these figures are the same as those used in figure 4.

This random variable is well defined for the unclipped population. Equation (36) is plotted in figure 9. In the limit $\tilde{\mu}T \ll 1$, $w_0(\tau)$ for the DMI process has the asymptotic behaviour

$$w_0(\tau) = a\left(\frac{\eta}{\tilde{\mu}}\right)^{\nu}(\tilde{\mu}\tau)^{-(1-\nu)} = a\frac{(\eta\tau)^{\nu}}{\tilde{\mu}\tau} \tag{37}$$

but for large times the emissions become uncorrelated and the tail of the distribution is exponential, as expected. The probability density to the first count for the BDMI process has similar asymptotic behaviour, except that $\tilde{\mu}$ is replaced by $\tilde{\mu} - \lambda$.

The mean time to the first count exists and is given by

$$\langle\tau_0\rangle = \int_0^{\infty}(1 - \bar{c}(\tau))\,\mathrm{d}\tau$$

and the principal parameter affecting this statistic is the index ν. Figure 10 shows a plot of the mean time to the first count as a function of the index ν. The relationship is predominantly linear for increasing values of ν.

The probability density of the time τ between counts, can be constructed as [15]

$$w_1(\tau) = \frac{\tau}{\bar{c}(\tau)}\frac{\partial^2 Q_c(1;\tau)}{\partial\tau^2}. \tag{38}$$

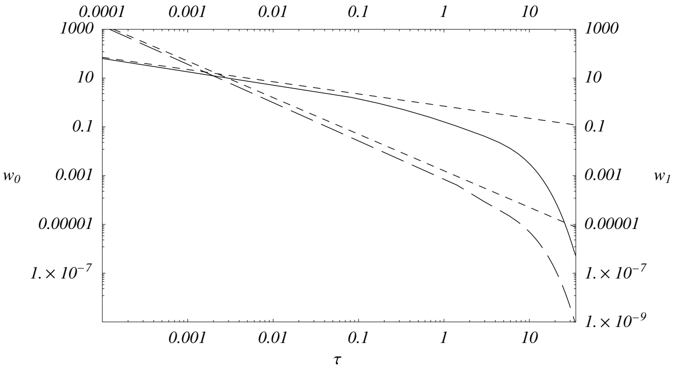

Figure 9. Probability density for the time to the first event and inter-event times of the counts shown by the full and long dashed lines respectively, together with their asymptotes shown by the short dashed lines. The parameters are as for figure 4 and the inner scale $\tau_0 = 0.000\,01$ [14].

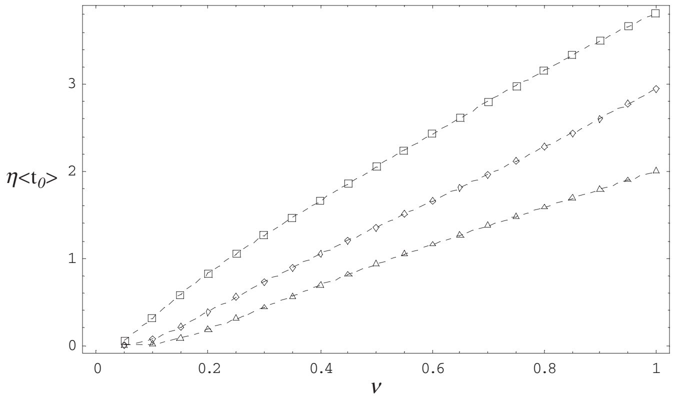

Figure 10. The mean time to the first counted event as a function of the index ν, for values of $\tilde{\mu} = 0.01$ (△), $\tilde{\mu} = 2$ (◇), $\tilde{\mu} = 5$ (□).

This is valid for all processes where $\bar{c}$ scales linearly with τ, however, from equation (31), the scaling of $\bar{c}$ with τ implies that the apparent rate of occurrence of events $\bar{c}/\tau$ increases with increasing resolution. Therefore in order to define $w_1(\tau)$ for the DMI and BDMI processes, a minimum resolution time τ_i is required, thus

$$w_1(\tau) = \frac{1}{\dot{\bar{c}}(\tau_i)}\frac{\partial^2 Q_c(1;\tau)}{\partial\tau^2} = \frac{1}{w_0(\tau_i)}\frac{\partial^2 Q_c(1;\tau)}{\partial\tau^2} \qquad \text{for} \quad \tau_i \leqslant \tau < \infty$$
$$\approx \frac{(1-\nu)}{\tau_i}\left(\frac{\tau_i}{\tau}\right)^{2-\nu} \qquad \text{for} \quad \tau_i \leqslant \tau \ll 1. \tag{39}$$

Figure 9 illustrates the power-law behaviour of $w_1(\tau)$ for $\tau \ll 1$ and the subsequent exponential cut-off as τ increases. The mean inter-event time and its second moment can now be constructed:

$$\langle\tau_1\rangle = \frac{Q_c(1;\tau_i)}{w_0(\tau_i)} + \tau_i \qquad \tau \geqslant \tau_i$$
$$\frac{\langle\tau_1^2\rangle}{\langle\tau_1\rangle^2} = \frac{1}{\langle\tau_1\rangle^2}\left(\tau_i\left(\tau_i + 2\frac{Q_c(1;\tau_i)}{w_0(\tau_i)}\right) + \frac{2}{w_0(\tau_i)}\int_{\tau_i}^{\infty}(1-\bar{c}(\tau))\,\mathrm{d}\tau\right). \tag{40}$$

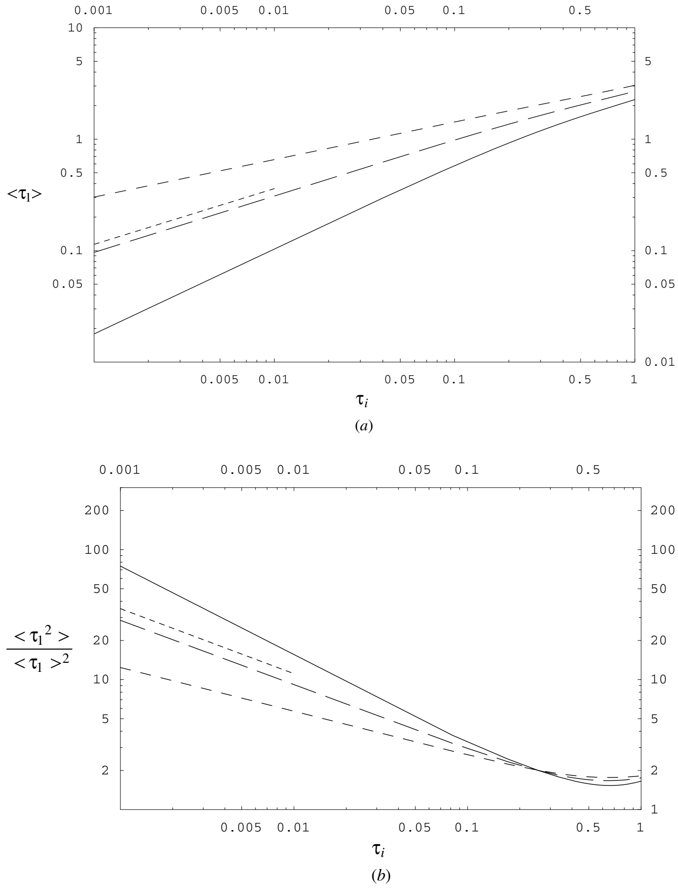

Figure 11. (*a*) The mean inter-event time as a function of the minimum resolution time. The full, large dashed and dashed lines represent values for $\nu = 1/4$, $1/2$ and $2/3$ respectively. (*b*) Variance of the inter-event times. The full, large dashed and dashed lines represent values for $\nu = 1/4$, $1/2$ and $2/3$ respectively. For both figures the small dashed line represents the asymptote for $\nu = 1/2$.

Figure 11(*a*) shows a plot of the mean inter-event time as a function of the resolution τ_i. As can be seen from the dashed line ($\nu = 1/2$) the mean shows fractal scaling at small values of the minimum resolution time of order

$$\langle \tau_1 \rangle \sim \frac{\tilde{\mu}}{a\eta^{\nu}} \tau_i^{1-\nu}. \tag{41}$$

Figure 11(*b*) shows a plot of the variance of the inter-event times as a function of the minimum resolution τ_i. The dashed line represents the asymptote for $\nu = 1/2$ with behaviour that scales like $(\eta\tau_i)^{-(1-\nu)}$. Note once more the power-law dependence for small values of $\eta\tau_i$.

5. Summary and discussion

This paper has investigated two discrete stochastic population processes that can be tailored to produce specific stationary states. The models considered are the death–multiple immigrant (DMI) model, which is a generalization of the immigrant pair population process [12] and the second is the birth—death–multiple immigrant (BDMI), for which the above is augmented by births. Selecting the rates at which the immigrants are introduced to the population enables a wide range of stationary states to be accessed by these processes. This paper has focused on generating discrete stable distributions that have a power-law asymptote $P_N \sim 1/N^{1+\nu}$ where $0 < \nu < 1$, so that the means of the distributions do not exist. The immigration rates for the DMI process depend on the parameter ν alone but those for the BDMI process are conditional on the birth and death rates. In both cases it is found that the order m of the immigrants have rates with power-law asymptote $\alpha_m \sim 1/m^{1+\nu}$, but it is important to stress that the rates are not a probability distribution, nor is the stationary probability distribution identical to these rates. The two processes are Markovian and so all joint properties of the processes are known including, in principle, the autocorrelation and higher order conditional statistical measures. However, due to the power-law nature of the distributions, none of the moments or correlations exist. This prompts the next aspect considered by the paper: how may such populations be monitored and characterized?

Monitoring the population was modelled by counting the number of emigrants that leave as a result of an additional death process. These emigrants form a series of successive events which are sampled for a period equivalent to the integration time of the detection scheme. Because the rate at which the emigrants leave is proportional to the instantaneous size of the internal population, the monitored population reflects this. Hence no moments or correlation properties can be defined for the integrated statistics of either model.

Any experimental monitoring method will saturate in excess of its dynamic range, thereby suppressing large fluctuations, or those occurring too rapidly. This can be idealized by hard limiting without losing the characteristic intermittent behaviour of the time series. Hard limiting reveals how the now defined correlation properties have a characteristic fluctuation time associated with them. Intermittency is best quantified by the probability density for the time between events, and this has a power-law regime at small times. However, all moments of the inter-event times exist because the fluctuations have a characteristic lifetime which introduces naturally an outer scale. A question that remains to be answered is how the accuracy of measurements of these quantities is affected by hard limiting and the duration of the experiment?

The models studied here produce the discrete analogue of the continuous stable probability distribution. The rates at which the immigrants appear were chosen specifically to produce stationary distributions of the discrete stable class. The useful property to 'tune' the stationary solution to have a particular form provides the means to study a wide range of fluctuation phenomena. These might include those that have power-law forms that nevertheless are not of the discrete stable class. Conversely, knowing the rates *a priori* enables the fluctuations of the ensemble to be deduced. Constructing appropriate statistics from the monitored populations then facilitates the important function of discriminating between candidate models of some complex process, and indeed characterizing scale-free fluctuations. In addition to providing a class of models with which to describe direct aspects of discrete or pulsed phenomena in complex systems, the work may be used to extend techniques in queuing theory. Another situation is if the population represents the number of coherent objects in a system. These may appear through nucleation or 'immigration'; give 'birth' to smaller structures, which then dissipate, and 'die', so providing a paradigm for various physical processes.

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Appendix. Clipped autocorrelation function

The autocorrelation function for the hard limited signal is

$$\langle c(t)c(t')\rangle = \sum_{c=0}^{\infty}\sum_{c'=0}^{\infty} cc' p(c, c') \tag{A.1}$$

which is unity when c and c' are both one or more, and zero otherwise. Hence

$$\langle c(t)c(t')\rangle = \sum_{c=1}^{\infty}\sum_{c'=1}^{\infty} p(c, c'). \tag{A.2}$$

Upon using elementary properties of the joint distribution $p(c, c')$ and the marginal distributions $p(c)$, one can show that

$$\langle c(t)c(t')\rangle = 1 - 2p(0) + p(0, 0) \tag{A.3}$$

where

$$p(0) = Q_{\text{cl}}(1; T) \qquad \text{and} \qquad p(0, 0) = R(1, 1; t, T). \tag{A.4}$$

Here, $R(z, z'; t, T)$ is the generating function for the joint distribution of counts in two non-overlapping equal time intervals and can be represented as follows,

$$R(z, z'; t, T) = Q_1(0, z'; T)Q_E(\Phi(0, z'; T); t - T)Q_1(f(\Phi(0, z'; T); t - T), z; T) \\ \times Q_E(\Phi(f(\Phi(0, z'; T); t - T), z; T); \infty)$$

where for the DMI process $f(s; t)$ and $Q_E(s; t)$ are given in equation (9) and $\Phi(s, z; T)$ and $Q_1(s, z; T)$ are given in equation (22). A full derivation for the definition of $R(z, z'; t, T)$ can be found in [16].

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